Int. J. Solids Structures Vol. 27, No. 4, pp. 423–434, 1991
Printed in Great Britain.

0020-7683/91 $3.00+.00

ON THE NONLOCAL ASPECT OF PROPAGATION OF THE ELASTIC GOODIER–BISHOP SURFACE WAVES†

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(*Received 27 September 1989; in revised form 24 January 1990*)

Abstract—Plane wave reflections at the free surface of a half-infinite elastic space are examined assuming long-range action of cohesive forces, and a grazing incidence of the waves. Using Kroener–Eringen constitutive equations, nonlocal elastic moduli are determined, and the difference–differential equations of motion for the displacement potential functions established. Satisfaction of the homogeneous boundary conditions determines the ratios of the amplitudes of the incident and reflected waves. A more detailed analysis of the grazing incidence of P-waves in a nonlocal medium confirms the existence of the Goodier–Bishop waves in the entire Brillouin frequency zone, at least for three different ranges of the particle interactions, and some experimental data available.

1. INTRODUCTION

As commonly known, the conventional theory of elasticity is remarkably successful in its predictions of the way material bodies respond to the action of external agents such as mechanical and electromagnetic forces and heat. Strangely enough, this is so notwithstanding the fact that the model of bodies accepted by the theory is such an abstraction as a plain continuous medium. And yet there are situations, few to be exact, in which the conclusions of the theory do not agree with experimental evidence, or come in conflict with our understanding of the real world.

Two relevant examples come to mind here. First, failure of classical elastodynamics to predict the dispersion of waves traveling in unbounded and semi-bounded media, a fact well established by actual observations. Second, the conclusion hardly compatible with the common sense that the stress concentrations at the tips of cracks and at the cores of dislocations become unbounded.

There were several endeavors in recent years of workers in mechanics to circumvent the difficulties encountered in the classical theory. One of these, of interest in the present note, gave birth to a new theory of elasticity, more general than the old one and called nonlocal.

The nonlocal theory differs from its classical counterpart in that it disputes the validity of the orthodox assumption that the cohesion forces binding the matter together are contact forces whose range of action is infinitesimal. Instead of this, the new theory accepts as relevant the firmly established findings of atomic physics, and asserts that the particle interactions are long-range forces that extend, at least in principle, over the entire body. It is important to note that due to the new approach the nonlocal theory is able to revise some of the questionable conclusions of the old theory; and so, for example, contrary to the above-mentioned classical prediction, the wave motion in nonlocal unbounded and semi-bounded media is found to be dispersive (Eringen, 1976; Nowinski, 1984a). Likewise, the singular points in cracks and dislocations become regular points (Ari and Eringen, 1983; Eringen, 1977), and by an appropriate definition of the concentrated force it is possible to predict a finite stress at the point of application of the force (Wang, 1988).

In the present note, we intend to shed some light, however limited, on another somewhat enigmatic question of existence of the undulatory motions known as Goodier–Bishop waves (Goodier and Bishop, 1952). The waves so named represent surface waves whose amplitude,

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strange as it is, increases (linearly) with an increasing distance from the surface. On account of this, the intensity of the waves at infinity becomes unbounded. This fact makes the phenomenon at odds with the natural St Venant requirement that a disturbance produced by the agencies acting in a finite domain should vanish at infinity. If this is so, one is tempted to suspect that perhaps the whole idea of the Goodier–Bishop waves has a purely formal character, and comes from the inadequacy of the postulational basis of the classical theory.

If such be the case, it seems both interesting and informative to inquire what can be said about the problem in question in the context of the nonlocal theory, that successfully clarifies other difficulties of the classical line of approach.

Proceeding in the desired direction, we first establish the nonlocal elastic moduli using the Eringen–Kroener form of the constitutive equations. We next arrive at the difference–differential equations of motion in terms of two displacement potential functions, and find the solution satisfying the boundary conditions of zero external tractions. A more detailed analysis of the grazing incidence of P-waves seems to confirm the reality (in the theory at least) of the Goodier–Bishop waves. This happens for three different ranges of cohesive interactions with due regard to the available experimental data.

It is worth noting that the Goodier–Bishop waves in their nonlocal aspect become possible for all admissible lengths in the Brillouin zone. In the classical theory, on the other hand, they exist only as very (infinitely) long waves.

2. AN AUXILIARY PROBLEM

As a preparation for the solution of the title problem, we examine the following auxiliary problem.

Let a plane longitudinal wave propage in an elastic infinite space with nonlocal properties in the direction of the x_1-axis of a Cartesian rectangular reference frame x_1, x_2, x_3. With the only identically non-zero displacement component,

$$u_1 = u_1(x_1, t), \tag{1}$$

there is associated, within the framework of a linear theory, the only non-zero strain component,

$$e_{11} = \partial u_1/\partial x_1. \tag{2}$$

We take the constitutive equation of the matter filling the space in the form proposed by Eringen (1972) on the basis of a general theory of constitutive equations. In the present case we have

$$\frac{\tau_{11}}{2\mu+\lambda} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{2\mu'(|x-x'|)+\lambda'(|x-x'|)}{2\mu+\lambda}\frac{\partial u_1'}{\partial x_1'}\,\mathrm{d}v', \tag{3}$$

where μ and λ are Lamé's constants, μ' and λ' are the nonlocal moduli (here kernel influence functions), $x \equiv (x_1, x_2, x_3)$ is the point under observation, x' a generic point of the medium, $\mathrm{d}v'$ the volume element of the medium, and the prime designates (here and later) quantities at the point x'. Inasmuch as the functions do not depend on the coordinates x_2' and x_3', then by the argument of Edelen in Nowinski (1984a, Appendix) there is no dependence on the coordinates x_2 and x_3, and the three-dimensional problem under discussion reduces to the one in one dimension. One may then write

$$\frac{\tau_{11}}{2\mu+\lambda} = \int_{-\infty}^{\infty} [2\mu'(|x_1-x_1'|)+\lambda'(|x_1'-x_1'|)]\frac{\partial u_1'}{\partial x_1'}\,\mathrm{d}x_1'. \tag{4}$$

By appeal to the Fourier exponential transform,

$$\bar{u}_1(k,t) = \int_{-\infty}^{\infty} u_1(x_1,t)\, e^{ikx_1}\, dx_1,$$

$$u_1(x_1,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{u}_1(k,t)\, e^{-ikx_1}\, dk, \tag{5}$$

and to the one-dimensional Faltung theorem we arrive at the equation

$$\frac{\tau_{11}(x_1,t)}{2\mu+\lambda} = -\int_{-\infty}^{\infty} ik\,\frac{2\bar{\mu}'(k)+\bar{\lambda}'(k)}{2\mu+\lambda}\,\bar{u}_1(k,t)\, e^{-ikx_1}\, dk. \tag{6}$$

The equation above introduced into the only identically non-vanishing equation of motion,

$$\frac{\partial \tau_{11}}{\partial x_1} - \rho\,\frac{\partial^2 u_1}{\partial t^2} = 0, \tag{7}$$

leads, by means of the separable representation

$$\bar{u}_1(k,t) = u_0(k)\, e^{i\omega t}, \tag{7a}$$

to the dispersion equation of the nonlocal continuum theory,

$$\frac{c^2}{c_1^2} = 2\pi\,\frac{2\bar{\mu}'(k)+\bar{\lambda}'(k)}{2\mu+\lambda}, \tag{8}$$

where $c = \omega/k$ is the actual wave speed in its nonlocal aspect, ω and k are the frequency and the wave number, respectively, and $c_1 = [(2\mu+\lambda)/\rho]^{1/2}$ is the speed of the longitudinal waves in an infinite space according to the conventional theory. Clearly, the nonlocal theory implies that the elastic waves depend on the wave number (respectively, on the wave length, $\Lambda = 2\pi/k$), so that the waves turn out to be dispersive. This agrees with the experimental evidence, but contradicts the conclusions of the classical elasticity. It should, of course, be recalled that the classical theory of elastic waves agrees essentially with observations as long as the wavelength is large enough as compared with the average atomic distance. As the wavelength decreases, effects of the discrete structure of bodies become more and more pronounced, and it is the objective of the nonlocal theory to account for these effects. To achieve this, a normal procedure of the nonlocal theory consists of establishing some kind of correspondence between the continuum theory and the atomic theory. There are several possibilities to secure the desired connection [see, e.g. Leibfried (1955, p. 185)]. One of these consists of the identification of the dispersion equation furnished by the nonlocal elastic theory [cf. eqn (8) above], which is assumed to be valid for waves of all lengths, with the corresponding equation derived in the Born–von Kármán atomic lattice dynamics. The last named equation retains its validity for waves of all lengths, and has the following form [see, e.g. Kittell (1967, p. 143)] for a monoatomic lattice:

$$\omega^2 = \frac{2}{M}\sum_{n=1}^{N} C_n(1-\cos nka). \tag{9}$$

Here M is the mass of an atom, a is the atomic spacing, and the C_ns are the constants determining the forces, with which the atom under observation (at the location $n = 0$) is acted upon by the planes of atoms removed by na. Setting $M = \rho a^3$, where ρ stands for the average atomic mass density, and combining eqns (8) and (9) gives

$$\frac{2\bar{\mu}(k)+\bar{\lambda}(k)}{2\mu+\lambda} = \frac{1}{2\pi}\left[\frac{1}{(2\mu+\lambda)a}\sum_{n=1}^{N} C_n n^2 \frac{\sin^2\left(\frac{nka}{2}\right)}{\left(\frac{nka}{2}\right)^2}\right]. \tag{10}$$

We note that the range of the wave number k is here as broad as

$$-\frac{\pi}{a} \leqslant k \leqslant \frac{\pi}{a}, \tag{10a}$$

and is being referred to as the first Brillouin zone.† This permits the length of the waves to vary between two atomic spacings, $2a$, and the infinity ($2a \leqslant \Lambda < \infty$) in contrast with the continuum theories in which k may tend to infinity and Λ to zero.

It is a straightforward matter to invert eqn (10) by means of a table of Fourier transforms [see, e.g., Magnus and Oberhettinger (1948)]. We then obtain

$$\frac{2\mu'(|x_1-x_1'|)+\lambda'(|x_1-x_1'|)}{2\mu+\lambda} = \frac{1}{2\mu+\lambda}\sum_{n=1}^{N} C_n \frac{n}{a^2}\left(1-\frac{|x_1-x_1'|}{na}\right) \tag{11}$$

where

$$|x_1-x_1'| < na, \tag{12}$$

and $n = 1, 2, \ldots, N$, respectively. Outside region (12), the right-hand side of eqn (11) becomes equal to zero. We note that the expression

$$\delta_S(x_1-x_1') \equiv \frac{1}{na}\left[1-\frac{|x_1-x_1'|}{na}\right] \tag{13}$$

in eqn (11) constitutes a term of a delta sequence‡ since the integral of δ_S taken over the interval $-\infty < x_1' < \infty$ is equal to 1, and moreover if $a \to 0$ then $\delta_S \to \infty$.

Suppose now that in (11) one sets $N = 1$ and writes the constitutive eqn (1) in the form appropriate for the case in question, namely

$$\tau_{11}(x_1, t) = \frac{C_1}{a}\int_{-\infty}^{\infty} \frac{\partial u(x_1', t)}{\partial x'} \delta_S(x_1-x_1')\,\mathrm{d}x_1'. \tag{14}$$

Inasmuch as for $a \to 0$ the term $\delta_S(x_1-x_1')$ tends to the delta function, $\delta(x-x')$ then by the well-known substitution property of the delta function the equation above tends to the limit

$$\tau_{11}(x_1, t) = \frac{C_1}{a}\frac{\partial u(x_1, t)}{\partial x_1}. \tag{15}$$

In the limit, however, the nonlocal theory transforms into its classical counterpart, so that eqn (15) expresses a trivial Hooke's law $\tau_{11} = (2\mu+\lambda)\,\partial u/\partial x_1$.

A comparison of the two preceding equations gives immediately that

† Confining the wave number to the first Brillouin zone removes the ambiguity in the wavelength corresponding to the given frequency.

‡ We recall that, speaking crudely, a delta sequence is a sequence of indexed functions the limit of which for the index tending to infinity is a Dirac delta function.

$$C_1 = (2\mu+\lambda)a \tag{16}$$

with μ and λ as Lamé's constants.

With all the preliminaries out of the way, we now focus on the examination of the title problem; this is done in the subsequent section.

3. GENERAL EQUATIONS

We now return to our title problem, and suppose that the surface of an elastic isotropic homogeneous and nonlocal half-infinite medium coincide with the x_1x_3-plane of a rectangular Cartesian coordinate system x_1, x_2, x_3, with the x_2-axis pointing towards the interior of the medium (Fig. 1).

Let a plane harmonic wave (a longitudinal, P-wave, say) propagating in the $\mathbf{k}/k$ direction $[|\mathbf{k}| = k \equiv (k_1, k_2)]$ strike the boundary $x_2 = 0$ at the angle e_1 (respectively e_2, for a transverse wave, SV). The trace of the wave front in the plane $x_3 = 0$ is L, and the reflected waves consist of the longitudinal, P′-wave, and the transverse, SV′-wave.

The constitutive equation of the nonlocal medium in the considered plane strain two-dimensional problem takes the form

$$\tau_{ij}(x,t) = \int_A [2\mu'(|x-x'|)e'_{ij}(x',t)+\lambda'(|x-x'|)e'_{kk}(x',t)\delta_{ij}]\,dx', \tag{17}$$

where $i,j = 1,2$, $x \equiv (x_1, x_2)$, τ_{ij} is the stress tensor, $e_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$ is the strain tensor, $u = (u_1, u_2)$ is the displacement vector, t is the time, and A is the area of the cross-section $x_3 = 0$ of the medium.

In order to pass from the one-dimensional problem discussed in Section 2 to the two-dimensional in hand, we wish to make the following assumptions.

We select the nonlocal moduli in a separable form using the designation

$$F_N(|x_i-x'_i|)\cdot G_N(|x_k-x'_k|), \tag{18}$$

where $i,k = 1,2$, and $i \neq k$.

Guided by the result (10) we seek the function F_N in a form similar to (11), namely

$$F_N(|x_i-x'_i|) = \sum_{n=1}^{N} C_n^{(\cdot)}\frac{n^2}{a}\left(1-\frac{|x_i, x'_i|}{na}\right), \tag{19}$$

where $i = 1,2$, and the coefficients $C_n^{(\cdot)}$ are denoted in the following by $C_n^{(\lambda)}$ or $C_n^{(\mu)}$ depending on whether the function F_N is associated with the modulus λ' or μ', respectively.

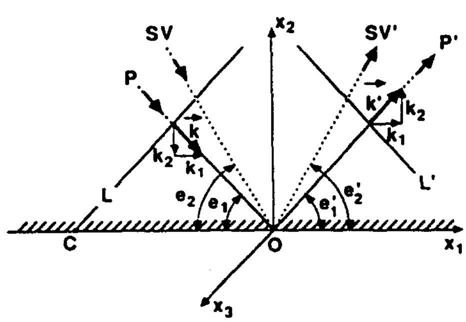

Fig. 1. Geometry of the problem.

We take advantage of the generality of the principle of attenuating neighborhood† and identify the function $G_S(|x_k-x'_k|)$ with a term $\delta_S(x_k-x'_k)$ of an appropriate chosen delta sequence.

As regards the functions of the sequence we adopt the simplifying assumption that the functions possess the familiar shifting property characteristic of the Dirac function; as a consequence, for any sufficiently regular function $f(x'_i)$ there is

$$\int_{-\infty}^{\infty} f(x'_i)\,\delta_S(x_i-x'_i)\,\mathrm{d}x'_i \approx f(x_i), \tag{20}$$

with $i = 1, 2$ and $S = 1, 2, 3, \ldots$. This relation becomes, of course, the more exact the higher the value of the index S.

Finally, we decide to associate in expression (18) a pair of indices $i = 1$, $k = 2$ ($i = 2$, $k = 1$) with the increments of the displacements in the x_1-direction (x_2-direction).

This brings us back at last to the main line of the argument. We invoke the standard representation of the displacements in terms of a pair of potential functions $\varphi(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$,

$$u_1 = \varphi_{,1}+\psi_{,2}, \quad u_2 = \varphi_{,2}-\psi_{,1}. \tag{21}$$

Substitution of the above into the equations of motion yields

$$\int_A \frac{\lambda'+2\mu'}{\lambda+2\mu}(\phi'_{,11}+\phi'_{,22})\,\mathrm{d}x'_1\,\mathrm{d}x'_2 = \frac{1}{c_1^2}\ddot{\phi}, \tag{22}$$

and a similar equation for the function ψ, provided $c_1^2 = (\lambda+2\mu)/\rho$ is replaced by $c_2^2 = \mu/\rho$, and $(\lambda'+2\mu')/(\lambda+2\mu)$ by μ'/μ, respectively.

We combine eqns (21) with eqns (22), respectively, and arrive at the following difference-differential equations for the function ϕ:

$$\sum_{n=1}^{N} \frac{C_n^{(\lambda+2\mu)}}{(\lambda+2\mu)a^3}[\phi(x_1+na, x_2, t)+\phi(x_1, x_2+na, t)-4\phi(x_1, x_2, t)$$
$$+\phi(x_1-na, x_2, t)+\phi(x_1, x_2-na, t)] = \frac{1}{c_1^2}\ddot{\phi}(x_1, x_2, t), \tag{23}$$

and a similar equation for the function ψ with $(\lambda+\mu)$ replaced by μ, and c_1^2 by $c_2^2 = \mu/\rho$.

We cast the functions ϕ and ψ in the obvious forms,

$$\phi = A_1\,\mathrm{e}^{\mathrm{i}(k_1x_1-k_1\delta_1x_2-\omega t)}+A_2\,\mathrm{e}^{\mathrm{i}(k_1x_1+k_1\delta_1x_2-\omega t)},$$
$$\psi = B_1\,\mathrm{e}^{\mathrm{i}(k_1x_1-k_2\delta_2x_2-\omega t)}+B_2\,\mathrm{e}^{\mathrm{i}(k_1x_1+k_2\delta_2x_2-\omega t)}. \tag{24}$$

The functions above satisfy the governing equations such as (23), provided

$$\frac{\omega^2}{c_1^2}-\sum_{n=1}^{N}\frac{4C_n(\lambda+2\mu)}{(\lambda+2\mu)a^3}\left(\sin^2\frac{k_1na}{2}+\sin^2\frac{k_2na}{2}\right) = 0, \tag{25}$$

with $k_2 = k_1\delta_1$, and

† According to the principle proposed by Eringen (1962), and based on actual observations, the interactions between particles decay rapidly with an increasing distance between them. Practically, the range of cohesion forces may include 15 interatomic distances and be of the order of 10^{-7} cm.

$$\frac{\omega^2}{c_2^2} - \sum_{n=1}^{N} \frac{4C_n(\mu)}{\mu a^3}\left(\sin^2\frac{k_1 na}{2} + \sin^2\frac{k_2 na}{2}\right) = 0, \tag{26}$$

with $k_2 = k_1\delta_2$. It is easily verified that

$$\tan e_1 = -\delta_1, \quad \tan e_2 = -\delta_2, \quad \tan e_1' = \delta_1, \quad \tan e_2' = \delta_2, \tag{27}$$

(cf. Fig. 1), and that eqns (25) and (26) reduce to the classical equations if $a \to 0$, $n = 1$, $C_1^{(\lambda+2\mu)} = (\lambda+2\mu)a$, and $C_1^{(\mu)} = \mu a$; that is, if the nonlocal medium becomes converted into the conventional, local, one [see, e.g. Ewing *et al.* (1957, p. 26)]. In this case also the coefficients of the closest neighboring interactions, $C_1^{(\lambda+2\mu)}$ and $C_1^{(\mu)}$, reduce to the corresponding Lamé's constants (according to the already mentioned correspondence principle of the lattice dynamics [see, e.g. Kittel (1967, p. 147); Eringen (1962, p. 248)].

With all the above in mind we write the equations of the stress components of interest, τ_{12} and τ_{22} as

$$\tau_{12}(x_1, x_2, t) = \sum_{n=1}^{N} \frac{4}{a^3} C_n^{(\mu)} \left\{ (A_1\phi_1 - A_2\phi_2)\left(\delta_1 \sin^2\frac{k_1 na}{2} + \frac{1}{\delta_1}\sin^2\frac{k_1\delta_1 na}{2}\right) + (B_1\psi_1 + B_2\psi_2)\left(\sin^2\frac{k_1 na}{2} - \sin^2\frac{\delta_2 k_1 na}{2}\right)\right\},$$

$$\tau_{22}(x_1, x_2, t) = -\sum_{n=1}^{N} \frac{4}{a^3} \left\{ C_n^{(\mu)}(A_1\phi_1 + A_2\phi_2)\left(\sin^2\frac{\delta_2 k_1 na}{2} - \sin^2\frac{k_1 na}{2}\right) - (B_1\psi_1 - B_2\psi_2)\left(C_n^{(\lambda)}\delta_2 \sin^2\frac{k_1 na}{2} - \frac{1}{\delta_2} C_n^{(\lambda+2\mu)} \sin^2\frac{\delta_2 k_1 na}{2}\right)\right\}, \tag{28}$$

where

$$\varphi_1 = e^{i(k_1x_1 - k_1\delta_1x_2 - \omega t)}, \quad \varphi_2 = e^{i(k_1x_1 + k_1\delta_1x_2 - \omega t)},$$
$$\psi_1 = e^{i(k_1x_1 - k_1\delta_2x_2 - \omega t)}, \quad \psi_2 = e^{i(k_1x_1 + k_1\delta_2x_2 - \omega t)}. \tag{29}$$

In preparation for the discussion of the Goodier-Bishop waves we now assume that the surface of the half-space is free from external load, so that

$$\tau_{12} = 0, \quad \tau_{22} = 0 \quad \text{at} \quad x_2 = 0, \tag{30}$$

for any value of the coordinate x_1 and at all times t. To save on non-essential calculations that tend to obscure rather than clarify the main points of the problem, we confine our attention to the case in which the incident wave represents a P-wave. Then $B_1 = 0$ (thus excluding the incidence of an SV-wave), and the ratios of the amplitudes of the incident and reflected (P′ and SV′) waves turn out to be

$$\frac{A_2}{A_1} = \frac{\Gamma_1\Gamma_3 + \Gamma_2^2}{\Gamma_1\Gamma_3 - \Gamma_2^2}, \quad \frac{B_2}{A_1} = \frac{2\Gamma_1\Gamma_2}{\Gamma_1\Gamma_3 - \Gamma_2^2}, \tag{31}$$

where

$$\Gamma_1 = \sum_{n=1}^{N} C_n^{(\mu)}\left(\delta_1 s_n + \frac{1}{\delta_1}s_{1n}\right), \quad \Gamma_2 = \sum_{n=1}^{N} C_n^{(\mu)}(s_n - s_{2n}), \quad \Gamma_3 = \sum_{n=1}^{N}\left[C_n^{(\lambda)}\delta_2 s_n - C_n^{(\lambda+2\mu)}\frac{1}{\delta_2}s_{2n}\right],$$

$$s_{1n} = \sin^2\frac{k_1\delta_1 na}{2}, \quad s_{2n} = \sin^2\frac{k_1\delta_2 na}{2}, \quad s_n = \sin^2\frac{k_1 na}{2}. \tag{32}$$

We note that: (a) if $N = 1$, that is, if the particle interactions are limited to the closest neighbors, then relations (31) become identical with those derived directly in Nowinski [1984b, eqns (28)–(30)]; (b) if in addition one makes the interatomic distance $a \to 0$, and the interaction parameters C_n convert into the Lamé constants, then relations (31) transform into those well-known from the classical theory [see, e.g. Ewing *et al.* (1957), eqns (2.11)].

4. GOODIER–BISHOP SURFACE WAVES

Before starting a discussion of the Goodier–Bishop waves in more detail, it is worth mentioning the unwelcome fact that for grazing incidence [as follows from the inspection of eqns (31)], that is, for $\delta_1 = -\tan e_1 = 0$, it follows that $A_2 = -A_1$ and $B_2 = 0$, and eqns (24) cease to represent any motion at all. It was Goodier and Bishop (1952) who showed that application of a special limiting process† reveals the existence of the trains of waves even in this rather extreme state of affairs. We will not appeal to this process when examining the phenomenon from the nonlocal point of view. In lieu of this, we apply a simpler procedure proposed later by Jardetzky (1952) [see also Ewing *et al.* (1957, p. 30)].

To do this, we return to the case of incidence of a P-wave, and try the solution of eqn (23) in a more general form,

$$\phi(x_1, x_2, t) = f(x_2)\,\mathrm{e}^{i(k_1 x_1 - \omega t)}. \tag{33}$$

This assumption leads to the following difference equation which has to be satisfied by the function $f(x_2)$:

$$\sum_{n=1}^{N} \frac{C_n^{(\lambda+2\mu)}}{(\lambda+2\mu)a^3}\left[f(x_2+na)+f(x_2-na)-2f(x_2)-4\sin^2\frac{k_1 na}{2}\right]+\frac{\omega^2}{c_1^2}=0. \tag{34}$$

We now pose $f(x_2) = \mathrm{e}^{i\alpha x}$, and easily find the associated characteristic equation

$$\sum_{n=1}^{N} \frac{4C_n^{(\lambda+2\lambda)}}{(\lambda+2\mu)a^3}\left[\sin^2\frac{\alpha na}{2}+\sin^2\frac{k_1 na}{2}\right]=\frac{\omega^2}{c_1^2}, \tag{35}$$

where, as before, c_1 denotes the velocity of longitudinal waves in their local aspect. We recall that for any harmonic wave there is

$$\frac{\omega}{k} = c, \tag{36}$$

where c is the phase velocity and $\mathbf{k}$ the wave vector. We then refer to eqns (1.8) and (1.11) in Nowinski (1989), and conclude that the ratio of the squares of velocities of the longitudinal waves in the nonlocal and local case is

$$\frac{c_1^2\,\mathrm{nonloc}}{c_1^2} = \sum_{n=1}^{N} \frac{C_n^{(\lambda+2\mu)}n^2}{(\lambda+2\mu)a}\,\frac{\sin^2\dfrac{nka}{2}}{\left(\dfrac{nka}{2}\right)^2}. \tag{37}$$

† According to the *ad hoc* assumption of Goodier and Bishop, the product $A_1 e_1$ is to remain constant when $e_1 \to 0$ and implicitly $A_1 \to \infty$.

Keeping the three preceding equations, and the familiar trigonometric relation ($i = \sqrt{-1}$, β real)

$$\sin i\beta = i \sin h\beta, \tag{37a}$$

in mind we examine the following particular cases.

4.1. *Classical case*

In this case one has to set $N = 1$, and make the lattice parameter a tend to zero. By appealing to eqns (35) and (36) one then finds that

$$\alpha_{1,2} = \pm k\left(\frac{c^2}{c_1^2} - 1\right)^{1/2}. \tag{38}$$

Since c is the velocity (directed along the x_1-axis) of the point of intersection of the wave front L with the x_1-axis (see Fig. 1), then eqn (38) implies that $\alpha = \pm \tan e_1$. This result coincides, as to be expected, with its classical counterpart to be found, for example, in Ewing *et al.* (1957, p. 26). Inasmuch as for a grazing incidence there evidently is $c = c_1$, then the root of the characteristic eqn (35) becomes a double root equal to zero. On account of this, there is

$$f(x_2) = D_1 e^{\alpha_1 x_2} + D_2 x_2 e^{\alpha_2 x_2} = D_1 + D_2 x_2, \tag{38a}$$

and we conclude that a longitudinal P-wave moving along the interface of a classical, local, half-space produces, in fact, a reflected longitudinal P′-wave aimed in the same direction, but with the amplitude increasing linearly with increasing distance from the surface. A simultaneously generated reflected SV′-wave turns out to be of an ordinary kind. Both these facts characterize the wave motion predicted by Goodier and Bishop.

4.2. *Nonlocal case* N = 1

In this case we again set $N = 1$, but appeal to eqn (37) which, for $N = 1$, simplifies to

$$c_{1\,\mathrm{nonloc}} = c_1 \frac{\sin \frac{ka}{2}}{\frac{ka}{2}}, \tag{39}$$

[compare Nowinski (1989, eqn (1.13))]. Likewise, (35) becomes

$$\sin^2 \frac{\alpha a}{2} = \frac{a^2 k^2}{4}\left(\frac{c^2}{c_1^2} - \frac{\sin^2 \frac{k_1 a}{2}}{\left(\frac{k_1 a}{2}\right)^2}\right) \tag{40}$$

and for grazing incidence, on account of relation (39) and the fact that in the present case $k = k_1$, we get

$$\sin^2 \frac{\alpha a}{2} = 0. \tag{40a}$$

Table 1. Force constants $C_n^{(\lambda+2\mu)}$

C_n (10^3 dynes cm^{-1})	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
Lead	14.8	4.9	−1.34	1.27	−0.98	0.45	−0.18	0.27	0.14
Aluminum [100]	38.6	1.15	1.30	−0.65	0.04	−0.10	−0.04	0.05	
Aluminum [110]	15.2	24.3	−1.82	−0.78	1.22	0.04	0.39	−0.05	

This condition may be satisfied only if $\alpha = 0$, irrespective of whether α is considered real or imaginary. In fact, in the first case we may argue that the parameter α has the character of a wave number. If this is so, and if one wishes to remain in the first Brillouin zone, then one is able to demand that $\alpha \leqslant \pi/a$ and $\alpha a/2 \neq n\pi$ for $n = 1, 2, \ldots$, so that consequently $\alpha = 0$. This brings us back to the Goodier–Bishop solution, and excludes other possible courses. On the other hand, if one treats α as imaginary, then eqn (37a) implies immediately that $\alpha = 0$ and one recovers once more the Goodier–Bishop case.

It is now of special interest to determine whether by taking higher values of the index N than 1, it is possible to arrive at conclusions differing from those arrived at in our previous discussion. We illustrate this point by examining successively the cases $N = 2$ and $N = 3$.

4.3. *Nonlocal case* N = 2

Let us first assume that $N = 2$. A simple manipulation furnishes the equation

$$\sin^2\frac{\alpha a}{2}\left[\sin^2\frac{\alpha a}{2} - \left(1 + \frac{C_1}{4C_2}\right)\right] = 0, \tag{41}$$

so that either

$$\sin^2\frac{\alpha a}{2} = 0, \tag{41a}$$

and one comes back to the Goodier–Bishop case, or there is

$$\sin^2\frac{\alpha a}{2} = 1 + \frac{C_1}{4C_2}. \tag{41b}$$

Whereas, in principle, there is no reason why the foregoing relation may be called in question, some of the available experimental data make such a conclusion less free from doubt. As an illustration, Table 1 displays the values of the force constant $C_n^{(\lambda+2\mu)}$ for two materials, with markedly different properties, namely lead† and aluminum‡ (for the latter, in two crystal directions [100] and [110]).

A glance at Table 1 convinces us that the constants $C_1^{(\lambda+2\mu)}$ and $C_2^{(\lambda+2\mu)}$ turn out to be positive. If this is the case, then irrespective of whether α is real or imaginary constraint (41b) is void. Consequently, if the particle interactions are restricted to the second closest neighbor, then the nonlocal theory confirms the existence of Goodier–Bishop waves, and excludes the occurrence of other undulatory motions.

† $\rho = 11.3$ and $c_1 = 1230$ m s^{-1}. C_n is calculated from the graph given in Fig. 15 in Kittel (1967) with inaccuracy likely to occur in the second digit.

‡ $\rho = 2.7$ and $c_1 = 5100$ m s^{-1}. c_n calculated from Fig. 4 in Yarnell and Warren (1965) giving ϕ_n from the equation

$$\omega^2 = \frac{1}{\rho a^3}\sum_{n=1}^{N}\phi_n(1-\cos nka).$$

4.4. *Nonlocal case* N = 3

As a final example let us consider the case corresponding to $N = 3$. A straightforward calculation yields the equation

$$\sin^2\frac{\alpha a}{2}\left[16C_3\sin^4\frac{\alpha a}{2}-4(C_3+6C_2)\sin^2\frac{\alpha a}{2}+(C_1+4C_2+9C_3)\right]=0. \tag{42}$$

In this case, either $\alpha = 0$, and we recover the Goodier–Bishop solution, or $\alpha \neq 0$ and the remaining biquadratic equation has the solution

$$\sin^2\frac{\alpha a}{2}=\frac{C_2+6C_3\pm[C_2^2-4C_3(C_1+C_2)]^{1/2}}{8C_3}. \tag{43}$$

If we now assume as we did before, that $C_1, C_2 > 0$ and add the condition that $C_3 < 0$ (see Table 1), then the discriminant in (43) becomes greater than zero.

The upper sign in (43) then gives $\sin^2 \alpha a/2 < 0$, and for real α constraint (43) becomes meaningless. On the other hand, if one assumes α to be imaginary, then there is a possibility of a wave motion whose amplitude varies exponentially with an increasing distance from the plane $x_2 = 0$. This solution corresponds to the wave motions of the already well known local (and recently discovered nonlocal) types named after Rayleigh [see Eringen (1973)] and Love [see Nowinski (1984a)]. As regards the lower sign in eqn (43), it is not difficult to convince oneself that both for real and imaginary α eqn (43) has no sense (at least as regards the data given in Table 1).

4.5. *Nonlocal general case*

In this case the range of cohesive forces remains unrestricted.

We first rearrange eqn (35) to read

$$\sum_{n=1}^{N}\frac{4C_n'^{(\lambda+2\mu)}}{(\lambda+2\mu)a^3k^2}\sin^2\frac{\alpha na}{2}=\frac{c^2}{c_1^2}-\sum_{n=1}^{N}\frac{4C_n'^{(\lambda+2\mu)}}{(\lambda+2\mu)a^3k^2}\sin^2\frac{kna}{2}. \tag{44}$$

We note that as before a grazing incidence of a P-wave implies that $c = c_{1\,\mathrm{nonloc}}$ and $k = k_1$, so that with due regard to eqn (37) the equation above simplifies to

$$\sum_{n=1}^{N}C_n^{(\lambda+2\mu)}\sin^2\frac{\alpha na}{2}=0. \tag{45}$$

One obvious solution here is $\alpha = 0$ corroborating the findings of Goodier and Bishop. Other solutions call for a detailed analysis of the heavily transcendental eqn (44). It need hardly be added that if one retains the requirement that the parameter α be confined to the first Brillouin zone, then at the right end of this zone we have

$$C_1^{(\lambda+2\mu)}+C_3^{(\lambda+2\mu)}+\cdots+C_{2n-1}^{(\lambda+2\mu)}=0, \tag{46}$$

where $(2n-1) \leqslant N$. Since the equation just written cannot be satisfied for both lead and aluminum, one has to conclude that: (a) in general, existence of wave motion other than that of the Goodier–Bishop type is questionable; (b) moreover, if such a motion actually exists it may not in general take place in the high frequency range corresponding to the right-hand end of the Brillouin zone. So far our analysis has involved the effects of longitudinal waves. All of previous arguments, however, with non-essential changes apply to the effects of transverse waves, and to save on space, we refrain from setting them down here.

5. CONCLUSIONS

The following conclusions summarize the main points of the present study:

(a) The nonlocal theory confirms the findings of the classical approach and finds the grazing incidence of a P-wave (respectively, an SV-wave) on a free surface of a half-space to produce a reflected Goodier–Bishop P-wave. This state of affairs is observed at least according to some of the available experimental data, and in media in which the cohesion reaches up to the third closest particle.

(b) In the last named case, the nonlocal theory predicts an additional generation of the waves of the Rayleigh and Love type not anticipated by the classical theory.

(c) The conclusions above relate to waves in question of any length (from infinitely long to as short as twice the interatomic distance) but it is doubtful whether high frequency (i.e. very short) waves of this kind may appear in media with particle interactions far distant in space. All this is true within the framework of the nonlocal theory, since—we repeat—the validity of the classical theory is limited to the range of very (infinitely) long waves.

(d) Taking all said into consideration, it is fair to conclude that the nonlocal model of elastic behavior does not contradict the phenomenon established by Goodier and Bishop, and seems to substantiate the fact that Goodier–Bishop's findings are not necessarily a strictly formal product of some inadequacies of the postulational basis of the classical theory of elasticity.

Acknowledgement—The author thanks his wife, Maria, for her cooperation in preparation of this paper.

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